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Local entropy production in turbulent shear flows: a high-Reynolds number model with wall functions

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Abstract

Entropy production in incompressible turbulent shear flows of Newtonian fluids is analysed systematically and incorporated into a CFD code. There are four different mechanisms of entropy production: dissipation in a mean and fluctuating velocity field and heat flux in a mean and fluctuating temperature field. Based on asymptotic considerations wall functions for the four production terms are developed. These wall functions are mandatory when high-Reynolds number turbulent models are used since peak values of entropy production occur in the immediate vicinity of a wall. As an example pipe flow with heat transfer is analysed and compared to results from a direct numerical simulation with special emphasis on the entropy production in the near wall region.

Keywords: Entropy production; Dissipation; Turbulent flow; Asymptotic considerations; CFD; k–ε model; Wall-functions; Second Law analysis

1. Introduction

Numerical prediction of heat transport phenomena in turbulent shear flows has attracted considerable attention over the past few decades. Modelling of these flows has come to a stage where pressure drop and heat transfer results are accurate even in complex geometries. Thus, computational fluid dynamics (CFD) has become state of the art in thermal engineering like in heat-exchanger design. However, all these CFD models only take into account the first law of thermodynamics.

An efficient use of energy is one of the major objectives in designing modern thermal systems like compact heat exchangers and power plants. This, however, can only be achieved if also the second law of thermodynamics is accounted for, since the amount of available work (also called *exergy*) is linked to the amount of entropy production, see [1]. Therefore, a thermal apparatus producing less entropy by irreversibilities destructs less available work (producing less *anergy*). This increases the total efficiency of a thermal system. The amount of entropy produced can be used directly as an efficiency parameter of the system, see [2–4].

Second law and entropy production analysis in particular have been widely used to evaluate the sources of irreversibilities in components and systems. The majority of these studies, however, are limited to a global analysis. The evaluation of local sources of irreversibilities, i.e. local entropy production, are often based on empirical correlations for the velocity and temperature fields. They, however, are known only for a small range of boundary conditions, see [5–8]. Other studies about local entropy production rates are for laminar problems only, see [9–15] or do not include the influence of solid walls in turbulent problems, see [16].

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Nomenclature

c_p	specific heat capacity, (J/(kg K))
Ec_τ	turbulent Eckert number $Ec_\tau = u_\tau^2/(c_p T_\tau)$, (–)
k	turbulent kinetic energy, (m^2/s^2)
k_Θ	variance of temperature fluctuations $k_\Theta = \overline{T'^2}/2$, (K^2)
$\vec{q}$	heat flux density vector, (W/m^2)
Pr	Prandtl number $Pr = c_p \cdot \mu/\lambda$, (–)
Pr_t	turbulent Prandtl number, (–)
Re	Reynolds number $Re = u_m \cdot D/\nu$, (–)
s	entropy per unit mass, (J/(kg K))
$\dot{S}_{PRO,\overline{D}}$	entropy production rate by viscous dissipation, (W/(m^3 K))
$\dot{S}_{PRO,D'}$	entropy production rate by turbulent dissipation, (W/(m^3 K))
$\dot{S}_{PRO,\overline{C}}$	entropy production rate by heat transfer with mean temperature gradients, (W/(m^3 K))
$\dot{S}_{PRO,C'}$	entropy production rate by heat transfer with gradients of the fluctuating temperature, (W/(m^3 K))
t	time, (s)
T	temperature, (K)
T_τ	friction temperature $T_\tau = -q_w/(\varrho c_p u_\tau)$, (K)
T_Φ	turbulent dissipation rate, (W/m^3)
u	velocity in x-direction, (m/s)
u_τ	skin friction velocity, $u_\tau = \sqrt{\tau_w/\varrho}$, (m/s)
v	velocity in y-direction, (m/s)
w	velocity in z-direction, (m/s)
x	coordinate in x-direction, (m)
y	coordinate in y-direction, (m)
z	coordinate in z-direction, (m)

Greek symbols

α	thermal diffusivity, (m^2/s)
ε	dissipation rate of turbulent kinetic energy, (W/kg)
ε_Θ	dissipation of the variance of temperature fluctuations, (K^2/s)
Φ	viscous dissipation of mechanical energy, (W/m^3)
Φ_Θ	entropy production term, (W K/m^3)
κ	von Karman's constant $K = 0.42$, (–)
κ_Θ	universal constant in the temperature field, $\kappa_\Theta = \kappa/Pr_t$, (–)
λ	thermal conductivity, (W/(m K))
μ	molecular viscosity, (kg/(m s))
ν	kinematic viscosity, $\nu = \mu/\varrho$, (m^2/s)
ν_T	turbulent eddy viscosity, (m^2/s)
ϱ	density, (kg/m^3)
τ	shear stress, (kg/(m s^2))
Θ^+	dimensionless temperature, $\Theta^+ = (T - T_w)/T_\tau$, (–)

Subscripts

$(\ )_{\ln}$	value in the logarithmic region
$(\ )_{mp}$	value at the midpoint
$(\ )_w$	value at the wall

Superscripts

$(\ )^+$	normalized by wall variables
$\overline{(\ )}$	time mean component
$(\ )'$	fluctuating component

Unfortunately, when designing a thermal apparatus important information inherent in the solution of the turbulent momentum and energy equations is never looked at nor used by the designers. However, it could be used to calculate the amount of entropy production and help the CFD engineer to improve the performance of his apparatus.

In this paper we present model equations for the calculation of the local entropy production in turbulent shear flows by extending the Reynolds-averaging procedure to the entropy equation. This equation serves to identify the entropy production sources, without need to solve the equation itself. In the basic equations including high-Reynolds number k–ε turbulence closure special attention has been given to the near wall regions where entropy production undergoes a steep change. Here, general wall functions for entropy production based on asymptotic considerations have been developed.

Adopting the presented model equations, local entropy production can be calculated in the post-processing phase of a CFD analysis. No further differential- or transport equation needs to be solved. Thus, the presented procedure does not require much CPU time and can easily be implemented in existing CFD codes. It is a tool to evaluate the performance of an apparatus in thermal engineering.

2. Transport equation for entropy

For a systematic derivation of a model for entropy production in turbulent flows, we start with the transport equation for entropy (Cartesian coordinates, incompressible fluid, single-phase flow, Fourier heat conduction), see [17]:

$$\varrho\left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z}\right) = -\mathrm{div}\left(\frac{\vec{q}}{T}\right) + \boxed{\frac{\Phi}{T}} + \boxed{\frac{\Phi_\Theta}{T^2}} \tag{1}$$

with the dissipation function Φ and Φ_Θ given in detail later (Eqs. (3) and (4)).

In Eq. (1) the two terms related to entropy production are marked by grey shaded boxes. The first one describes entropy production by viscous dissipation, the second one entropy production by heat transfer with finite temperature gradients. These terms are always positive and therefore act as a real source in Eq. (1). All other terms can be positive or negative depending on the direction of the flow and heat flux.

For example, a heat transfer apparatus with large areas of heat transfer encounters small temperature gradients and therefore small entropy production by heat transfer. However, due to the large area the pressure drop of this apparatus will be quite high, resulting in a large entropy production by dissipation. Since now both effects, heat transfer and pressure drop, have been linked to one single quantity (entropy production), the overall performance can be evaluated by the total entropy production, which should be as small as possible.

If we had not this single quantity, two completely different parameters would have to be brought together in order to find out if, for example, an increase of the heat transfer coefficient accompanied by an increase in pressure drop is an increase in the overall performance of the apparatus. This, however, would be like comparing apples and pears.

2.1. Time-averaging the transport equation for entropy

Eq. (1) is valid for the instantaneous values of entropy s, velocities u, v and w and temperature T. In the well known RANS (Reynolds averaged Navier Stokes) approach instantaneous values are split into time-mean and fluctuating parts, i.e. $s = \bar{s} + s'$, $u = \bar{u} + u', \ldots$.

These are inserted into Eq. (1) before it is time-averaged. In this time-averaging process additional turbulent terms emerge as will be shown in the following subsections.

2.1.1. Convective terms

On the left hand side of Eq. (1) after time-averaging for an incompressible fluid three additional terms appear:

$$\overline{\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z}} = \frac{\partial \bar{s}}{\partial t} + \bar{u}\frac{\partial \bar{s}}{\partial x} + \bar{v}\frac{\partial \bar{s}}{\partial y} + \bar{w}\frac{\partial \bar{s}}{\partial z} + \frac{\partial \overline{u's'}}{\partial x} + \frac{\partial \overline{v's'}}{\partial y} + \frac{\partial \overline{w's'}}{\partial z} \qquad (2)$$

2.1.2. Entropy production by dissipation

Time-averaging the entropy production by dissipation gives two groups of terms, one with mean and one with fluctuating quantities. They read:

$$\begin{aligned}\overline{\left(\frac{\Phi}{T}\right)} = \frac{\mu}{\overline{T}} \cdot & \left[2\left\{\left(\frac{\partial \bar{u}}{\partial x}\right)^2 + \left(\frac{\partial \bar{v}}{\partial y}\right)^2 + \left(\frac{\partial \bar{w}}{\partial z}\right)^2\right\}\right. \\ & \left. + \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)^2 + \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}\right)^2\right] \\ & + \frac{\mu}{\overline{T}} \cdot \left[2\left\{\overline{\left(\frac{\partial u'}{\partial x}\right)^2} + \overline{\left(\frac{\partial v'}{\partial y}\right)^2} + \overline{\left(\frac{\partial w'}{\partial z}\right)^2}\right\}\right. \\ & \left. + \overline{\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)^2} + \overline{\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)^2} + \overline{\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right)^2}\right].\end{aligned} \qquad (3)$$

Here T' in the denominator appears only in higher order terms when expanded into a series and therefore is neglected.

The first group of terms containing the mean velocity gradients can be interpreted as entropy production by dissipation in the mean flow field. This part of the dissipation is often referred to as *direct dissipation*. The second group of terms, containing the gradients of the fluctuating velocities hence is the entropy production by dissipation in the fluctuating part of the flow field. It is often called *indirect* or *turbulent dissipation*.

2.1.3. Entropy production by heat transfer

In the time-averaging process of the entropy production terms with respect to finite temperature gradients in Eq. (1) a factor $1/T^2$ appears. Again T' is neglected since it only appears in higher order terms. Entropy production due to finite temperature gradients then reads:

$$\begin{aligned}\overline{\left(\frac{\Phi_\Theta}{T^2}\right)} = & \frac{\lambda}{\overline{T}^2}\left[\left(\frac{\partial \overline{T}}{\partial x}\right)^2 + \left(\frac{\partial \overline{T}}{\partial y}\right)^2 + \left(\frac{\partial \overline{T}}{\partial z}\right)^2\right] \\ & + \frac{\lambda}{\overline{T}^2}\left[\overline{\left(\frac{\partial T'}{\partial x}\right)^2} + \overline{\left(\frac{\partial T'}{\partial y}\right)^2} + \overline{\left(\frac{\partial T'}{\partial z}\right)^2}\right]\end{aligned} \qquad (4)$$

Here, the first group of terms is the entropy production due to heat transfer with time-mean temperature gradients. The second group of terms is the entropy production by heat transfer due to fluctuating temperature gradients.

2.1.4. Time-averaged transport-equation for entropy

Summarising the time-averaging process of the transport Eq. (1) gives the following turbulent transport equation for the mean entropy:

$$\varrho\left(\frac{\partial\overline{s}}{\partial t}+\overline{u}\frac{\partial\overline{s}}{\partial x}+\overline{v}\frac{\partial\overline{s}}{\partial y}+\overline{w}\frac{\partial\overline{s}}{\partial z}\right)$$

$$=-\overline{\operatorname{div}\left(\frac{\vec{q}}{T}\right)}-\varrho\left(\frac{\partial\overline{u's'}}{\partial x}+\frac{\partial\overline{v's'}}{\partial y}+\frac{\partial\overline{w's'}}{\partial z}\right)$$

$$\underbrace{\boxed{+\frac{\mu}{\overline{T}}\left[2\left\{\left(\frac{\partial\overline{u}}{\partial x}\right)^2+\left(\frac{\partial\overline{v}}{\partial y}\right)^2+\left(\frac{\partial\overline{w}}{\partial z}\right)^2\right\}+\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)^2+\left(\frac{\partial\overline{u}}{\partial z}+\frac{\partial\overline{w}}{\partial x}\right)^2+\left(\frac{\partial\overline{v}}{\partial z}+\frac{\partial\overline{w}}{\partial y}\right)^2\right]}}_{\dot{S}_{PRO,\overline{D}}:=\text{entropy production by direct dissipation}}$$

$$\underbrace{\boxed{+\frac{\mu}{\overline{T}}\left[2\left\{\overline{\left(\frac{\partial u'}{\partial x}\right)^2}+\overline{\left(\frac{\partial v'}{\partial y}\right)^2}+\overline{\left(\frac{\partial w'}{\partial z}\right)^2}\right\}+\overline{\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)^2}+\overline{\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)^2}+\overline{\left(\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}\right)^2}\right]}}_{\dot{S}_{PRO,D'}:=\text{entropy production by turbulent dissipation}}$$

$$\underbrace{\boxed{+\frac{\lambda}{\overline{T}^2}\left[\left(\frac{\partial\overline{T}}{\partial x}\right)^2+\left(\frac{\partial\overline{T}}{\partial y}\right)^2+\left(\frac{\partial\overline{T}}{\partial z}\right)^2\right]}}_{\dot{S}_{PRO,\overline{C}}:=\text{entropy production by heat transfer with mean temperatures}}$$

$$\underbrace{\boxed{+\frac{\lambda}{\overline{T}^2}\left[\overline{\left(\frac{\partial T'}{\partial x}\right)^2}+\overline{\left(\frac{\partial T'}{\partial y}\right)^2}+\overline{\left(\frac{\partial T'}{\partial z}\right)^2}\right]}}_{\dot{S}_{PRO,C'}:=\text{entropy production by heat transfer with fluctuating temperatures}} \qquad (5)$$

Thus four groups of entropy production terms in turbulent flows can be identified in this systematic approach:

(1) $\dot{S}_{PRO,\overline{D}}$: entropy production rate by direct dissipation,
(2) $\dot{S}_{PRO,D'}$: entropy production rate by indirect (turbulent) dissipation,
(3) $\dot{S}_{PRO,\overline{C}}$: entropy production rate by heat conduction with mean temperature gradients,
(4) $\dot{S}_{PRO,C'}$: entropy production rate by heat transfer with fluctuating temperature gradients.

Other studies on local entropy production in turbulent flows often are incomplete, for example neglecting $\dot{S}_{PRO,C'}$, see [16].

As a consequence of the time averaging process new unknowns appear in the equations for the time mean quantities (closure problem). As far as entropy production is concerned they are $\dot{S}_{PRO,D'}$ and $\dot{S}_{PRO,C'}$ in Eq. (5). Note that Eq. (5) as a whole will not be solved but rather serves to identify all entropy production terms of the problem. Therefore turbulence modelling is needed only for these terms and will be provided in the following section.

3. Model equations for the local entropy production

After the four different entropy sources could be identified in the balance equation for the entropy we now want to model the two of them that are still unclosed ($\dot{S}_{PRO,D'}, \dot{S}_{PRO,C'}$). For that purpose information already available in a k–ε turbulence closure of the whole system of equations should be used as far as possible.

3.1. Entropy production by indirect (turbulent) dissipation

The entropy source group $\dot{S}_{PRO,D'}$ is closely related to the so-called turbulent dissipation rate T_Φ which appears in the k-equation of the k–ε model. It is

$$T_\Phi = \overline{T}\cdot\dot{S}_{PRO,D'} \qquad (6)$$

so that the k–ε model might provide the necessary information to determine $\dot{S}_{PRO,D'}$.

This model is a two-equation turbulence model based on the equations for the mechanical energy, k, of the velocity fluctuations and a corresponding equation for the dissipation of k, called ε-equation. In order to derive an approximation for the turbulent dissipation rate T_Φ we take a closer look at the transport equation for the mechanical energy of the velocity fluctuations (k-equation, with groups of terms abbreviated as T_{PV1}, T_{PV2}, T_{TD1}, T_{PRO}, T_{VD}, T_Φ, see [18] for details):

$$\varrho\left(\frac{\partial k}{\partial t}+\bar{u}\frac{\partial k}{\partial x}+\bar{v}\frac{\partial k}{\partial y}+\bar{w}\frac{\partial k}{\partial z}\right)$$
$$= T_{PV1}-T_{PV2}-T_{TD1}-T_{PRO}+T_{VD}-T_\Phi \qquad (7)$$

In Eq. (7) T_{PV1} and T_{PV2} are pressure/velocity correlations, T_{TD1} is the turbulent diffusion, T_{PRO} the production of turbulent kinetic energy, T_{VD} the viscous diffusion and T_Φ the turbulent dissipation. The dissipation rate ε which is used in all standard k–ε models, however, is not equal to the turbulent dissipation rate T_Φ/ϱ as one might expect. Instead, some terms of the group T_{VD} in the k-equation are combined with T_Φ to a quantity

$$\varepsilon = (T_\Phi - T_{VD} + \mu\Delta k)/\varrho \qquad (8)$$

which then appears in the k-equation. Here Δ is the Laplace operator.

The benefit is that now the two terms ($T_{VD}-T_\Phi$) in the k-equation of form (7) can be replaced by ($\mu\Delta k-\varrho\varepsilon$), a combination of terms that needs no explicit modelling since it is given in terms of k and ε already. This procedure is standard in all versions of the k–ε model. It should be kept in mind, however, that $\varrho\varepsilon$ is not the exact expression for the turbulent dissipation (which is T_Φ), and therefore in [18], for example, is named pseudo-dissipation. The difference between $\varrho\varepsilon$ and T_Φ, however, is asymptotically small, disappearing for $Re\to\infty$, see [19]. Furthermore, neither k nor ε can be determined exactly from the k- and ε-equations since they contain terms that again need modelling to yield a closed system of equations. Thus what finally has to be solved are

k- and ε-model-equations including a set of (empirical) coefficients.

To summarise: the approximations we introduced in connection with the turbulent dissipation term T_Φ are:

(1) Since T_Φ cannot be determined directly we replace it by $\varrho\varepsilon$ according to Eq. (8). Thus we assume that $\mu\Delta k - T_{VD}$ is small compared to T_Φ.
(2) Since ε cannot be determined exactly we replace the exact ε-equation by the ε-model-equation of the k–ε model.

Our model equation for the source term $\dot{S}_{PRO,D'}$ therefore is

$$\boxed{\dot{S}_{PRO,D'} = \frac{\varrho\varepsilon}{\overline{T}}} \tag{9}$$

where ε comes from the model-equation of the k–ε model.

3.2. Entropy production by fluctuating temperatures

If the entropy source $\dot{S}_{PRO,C'}$ should be determined in a way analogous to how we determined $\dot{S}_{PRO,D'}$ in Eq. (9), the turbulence model of the problem would have to be a four equation model for k, ε, k_Θ and ε_Θ.

Here k_Θ is the variance of the temperature fluctuations, $k_\Theta = \overline{T'^2/2}$ and ε_Θ is its dissipation. Such models exist, see for example [20], but they are not incorporated in standard CFD-codes. Nevertheless it is worthwhile to have a closer look at the k_Θ equation. It reads (see [18])

$$\varrho\left(\frac{\partial k_\Theta}{\partial t} + \bar{u}\frac{k_\Theta}{\partial x} + \bar{v}\frac{k_\Theta}{\partial y} + \bar{w}\frac{k_\Theta}{\partial z}\right) = T_{\Theta VD} - T_{\Theta TD} + T_{\Theta PRO} - \varrho\varepsilon_\Theta \tag{10}$$

Here $T_{\Theta VD}$ is a group of viscous diffusion terms, $T_{\Theta TD}$ is turbulent diffusion, $T_{\Theta PRO}$ production of k_Θ and $\varrho\varepsilon_\Theta$ its dissipation. The term ε_Θ in detail reads

$$\varepsilon_\Theta = \alpha\left[\overline{\left(\frac{\partial T'}{\partial x}\right)^2} + \overline{\left(\frac{\partial T'}{\partial y}\right)^2} + \overline{\left(\frac{\partial T'}{\partial z}\right)^2}\right] \tag{11}$$

with the thermal diffusivity α. It is closely related to the entropy production $\dot{S}_{PRO,c'}$ by

$$\varepsilon_\Theta = \frac{\overline{T}^2\dot{S}_{PRO,C'}}{\varrho c_p} \tag{12}$$

Since ε_Θ is not determined with a two equation k_Θ–ε_Θ model, we alternatively find it from the following considerations:

Except for regions very close to and very far from a wall, i.e. in the logarithmic region, there often is an equilibrium situation in which production and dissipation of k_Θ equal each other in magnitude, i.e. $\varrho\varepsilon_\Theta = T_{\Theta PRO}$, see Eq. (10). Here $T_{\Theta PRO}$ is

$$T_{\Theta PRO} = \varrho\left(-\overline{u'T'}\cdot\frac{\partial\overline{T}}{\partial x} - \overline{v'T'}\cdot\frac{\partial\overline{T}}{\partial y} - \overline{w'T'}\cdot\frac{\partial\overline{T}}{\partial z}\right) \tag{13}$$

If this term can be modelled we can relate it to ε_Θ and thus determine $\dot{S}_{PRO,C'}$.

Modelling of the terms $\overline{u_i'T'}$ in Eq. (13) now is based on the assumption of a constant turbulent Prandtl number $Pr_t = \nu_t/\alpha_t$ and a Boussinesque-like approach $-\overline{u_i'T'} = \alpha_t\partial\overline{T}/\partial x_i$ for the turbulent heat flux so that with $\nu_t = C_\mu k^2/\varepsilon$ we get

$$-\overline{u_i'T'} = \frac{\nu_t}{Pr_t}\frac{\partial\overline{T}}{\partial x_i} = \frac{C_\mu}{Pr_t}\frac{k^2}{\varepsilon}\frac{\partial\overline{T}}{\partial x_i} \tag{14}$$

Eq. (14) in (13) with $\varrho\varepsilon_\Theta = T_{\Theta PRO}$ gives

$$\varepsilon_\Theta = \frac{C_\mu}{Pr_t}\frac{k^2}{\varepsilon}\left[\left(\frac{\partial\overline{T}}{\partial x}\right)^2 + \left(\frac{\partial\overline{T}}{\partial y}\right)^2 + \left(\frac{\partial\overline{T}}{\partial z}\right)^2\right] \tag{15}$$

To summarise, the approximation we introduced in connection with the turbulent dissipation term ε_Θ are:

(1) Local equilibrium, i.e. $\varrho\varepsilon_\Theta = T_{\Theta PRO}$.
(2) A Boussinesque approach for $\overline{u_i'T'}$ and a constant turbulent Prandtl number.

Our model equation for the source term $\dot{S}_{PRO,c'}$ therefore is

$$\boxed{\dot{S}_{PRO,C'} = \frac{\alpha_t}{\alpha}\frac{\lambda}{\overline{T}^2}\left[\left(\frac{\partial\overline{T}}{\partial x}\right)^2 + \left(\frac{\partial\overline{T}}{\partial y}\right)^2 + \left(\frac{\partial\overline{T}}{\partial z}\right)^2\right]} \tag{16}$$

Eq. (16) shows that the *turbulent* entropy production by heat conduction, $\dot{S}_{PRO,C'}$, is closely linked to the *direct* entropy production by heat conduction. The only difference between these two terms is a factor α_t/α, which in regions far away from the wall can adopt values in the order of 100 or above.

An alternative approach, suggested by one of the referees of this paper, could be to link ε_Θ to the dissipation rate ε via $\varepsilon_\Theta = \varepsilon\, k_\Theta/k$. Then, however, an extended turbulence model is needed, including a transport equation for k_Θ.

4. A test case: local entropy production in a turbulent heated channel

So far, we have shown how all four sources of entropy production in turbulent flows can be calculated in a post-process. The information available in a turbulent heat transfer calculation with k–ε turbulence modelling is sufficient for an a posteriori entropy analysis. In this section some results and comparison with direct numerical simulation (DNS) are presented for a turbulent heated channel flow (our test case). The Reynolds number based on bulk velocity and channel width is $Re = 13981$. DNS results are available for Prandtl numbers $Pr = 0.71$ and $Pr = 5$, see [21]. In our calculations we use the standard k–ε model with wall functions and the same boundary conditions as in [21].

The direct entropy production rates for both, the DNS and the k–ε model calculations, are those in the RANS-equation for entropy, i.e. terms $\dot{S}_{PRO,\overline{D}}$ and $\dot{S}_{PRO,\overline{C}}$ in Eq. (5). The turbulent entropy production rates $\dot{S}_{PRO,D'}$, $\dot{S}_{PRO,C'}$ for the direct numerical simulations are calculated via Eqs. (9) and (12). In Fig. 1 the results for $Pr = 0.71$ are presented. The entropy production rates are non-dimensionalised by $u_\tau^2 \lambda/(\nu\alpha T_w^2/T_\tau^2)$. It turns out that for all Prandtl numbers for which results are available the entropy production rates calculated by the k–ε model equations show good agreement with the DNS calculations provided $y^+ > 50$, i.e. excluding the near wall region. All entropy production rates have peak values near the wall and especially for the production rates from mean gradients, i.e. $\dot{S}_{PRO,\overline{D}}$ and $\dot{S}_{PRO,\overline{C}}$, the model equation results are far off very close to the wall. This is due to the extremely steep gradients of mean velocity and temperature in the immediate vicinity of the wall. In the standard k–ε model they are accounted for by special wall functions. They are analytical expressions for the solutions in this wall nearest part of the flow field and exploit the universal nature of near wall turbulent physics. Thus, there is no need for an extremely fine grid that could resolve these steep gradients. Instead, the first finite volume of the numerical grid is rather large with the analytical solution incorporated.

However, such wall functions have not yet been established for the entropy productions terms, so that errors in these terms are extremely high in the wall nearest volume which is far too big for a resolution of extreme gradients. This is illustrated in Fig. 2 where the wall nearest finite volume extends to $y^+ = 27$. The DNS entropy production rate by direct dissipation (hatched area) by far exceeds that of the k–ε model equations (shaded area). Obviously without extra considerations in the wall adjacent volume entropy production rate calculations by the model equations result in inacceptable errors. And: most of the entropy is generated in the near wall region!

This clearly defines the next step: wall functions are needed for all four entropy production terms.

5. Wall functions for entropy production in turbulent shear flows

In order to obtain more reasonable values for the entropy production rates in the wall adjacent discrete

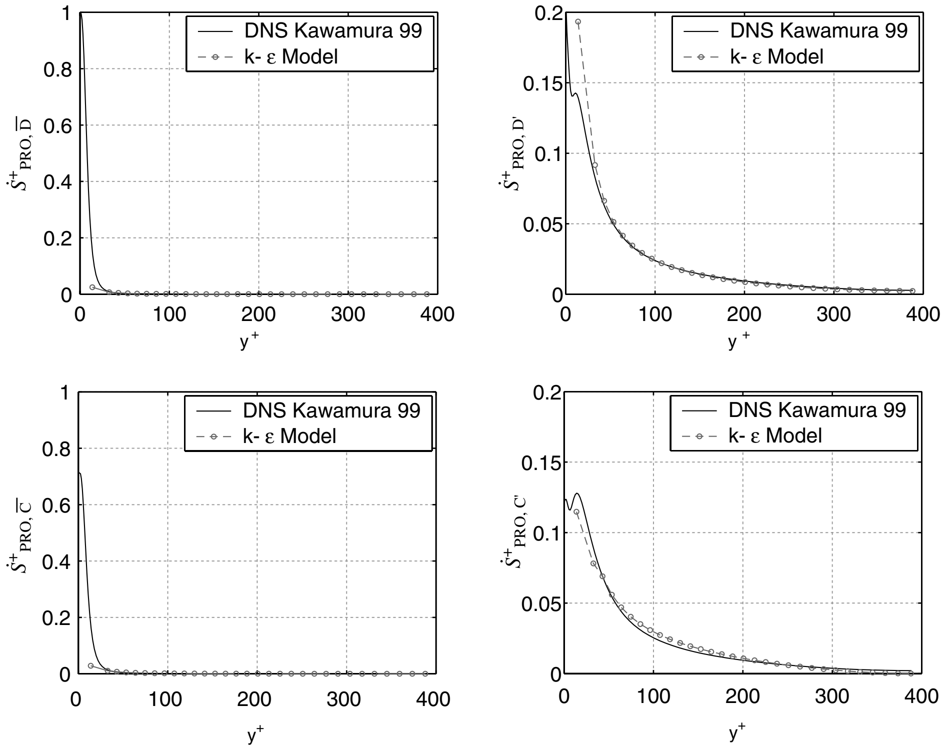

Fig. 1. Entropy production rates in turbulent heated channel flow from DNS results and $k-\varepsilon$ model equations, $Re = 13981$, $Pr = 0.71$.

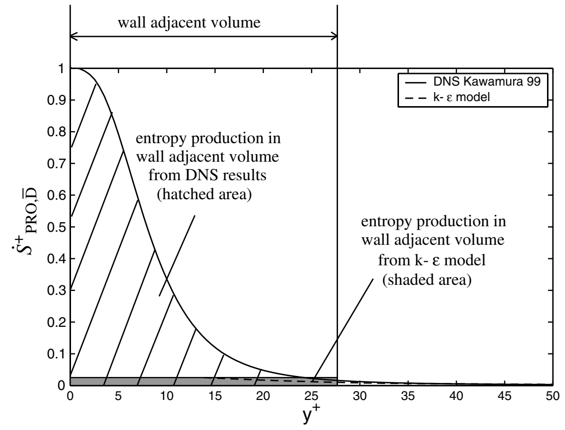

Fig. 2. Entropy production rates by direct dissipation of kinetic energy in the near wall region.

volumes in k–ε model calculations, wall function have to be implemented. These wall-functions should be analytical over the control volume so that a volume integrated value for the entropy production rate can be determined.

Wall-functions for the entropy production rates should be consistent with the asymptotic expansion of the velocity and temperature profiles for $Re \to \infty$. Based on these asymptotic expansions entropy production rates for $y^+ \to 0$ and $y^+ \to \infty$ can be found. Here, y^+ is the turbulent wall coordinate yu_τ/v with $u_\tau = \sqrt{\tau_w/\varrho}$ as skin friction velocity. With these asymptotes for $y^+ \to 0$ and $y^+ \to \infty$ the production rates in the whole y^+-range can be found approximately by a procedure proposed by Churchill and Usagi, see [22], provided one additional value for finite y^+ is known. This, however, should be a particular number for $Re \to \infty$ which cannot be extracted from DNS data, for example.

As an alternative approach we combine asymptotic and DNS considerations. We "construct" wall functions that are asymptotically correct for $y^+ \to 0$ and correspond to DNS results for finite values of y^+. Implicitly we thus assume that the universal nature of wall adjacent functions that is known to exist for $Re \to \infty$ is sufficiently developed for those Reynolds numbers already that can be reached by DNS calculations.

parallel velocity gradients, $(\partial \ldots/\partial x, \partial \ldots/\partial z)$, is (see Eq. (5))

$$\dot{S}^+_{PRO,\overline{D}} = \dot{S}_{PRO,\overline{D}} \cdot \frac{v\alpha\left(\frac{T_w}{T_\tau}\right)^2}{u_\tau^2 \lambda} = \frac{Ec_\tau \frac{T_w}{T_\tau}}{1 + \frac{T_\tau}{T_w}\Theta^+}\left(\frac{du^+}{dy^+}\right)^2 \tag{17}$$

DNS results suggest a $\dot{S}^+_{PRO,\overline{D}}$ -function of the general form

$$\dot{S}^+_{PRO,\overline{D}} = A_\tau \exp\left[-b_\tau (y^+ - a_\tau)^2\right] \tag{18}$$

with three constants (A_τ, b_τ, a_τ) left for a specific curve fit and asymptotic restraints, respectively.

From Eq. (17) we know

$$\dot{S}^+_{PRO,\overline{D}}(y^+ = 0) = Ec_\tau \frac{T_w}{T_\tau} \tag{19}$$

$$\left.\frac{d\dot{S}^+_{PRO,\overline{D}}}{dy^+}\right|_{y^+=0} = -Ec_\tau Pr \tag{20}$$

since $du^+/dy^+ = 1$ and $d\Theta^+/dy^+ = Pr$ at $y^+ = 0$. These two considerations as well as a third constraint (assumption about the location of the inflection point) gives

$$a_\tau = \frac{-9PrT_\tau y^{+2}_{\ln\tau}}{8T_w - 12y^+_{\ln\tau}PrT_\tau + 8\sqrt{T_w^2 - 3y^+_{\ln\tau}PrT_wT_\tau}} \tag{21}$$

$$b_\tau = \frac{8T_w - 12y^+_{\ln\tau}PrT_\tau + 8\sqrt{T_w^2 - 3y^+_{\ln\tau}PrT_wT_\tau}}{18T_w y^{+2}_{\ln\tau}} \tag{22}$$

$$A_\tau = Ec_\tau \frac{T_w}{T_\tau}\exp[b_\tau a_\tau^2] \tag{23}$$

with $y^+_{\ln\tau} = 11.6$ being the edge of the viscous sublayer. The semi-empirical wall-function for the entropy production by direct dissipation, (18), for two different Prandtl numbers is shown in Fig. 3 together with the DNS results from [21]. Including the asymptotic behaviour for $y^+ \to 0$ it correctly accounts for the change in the wall-gradient for higher Prandtl numbers.

This wall-function can be integrated so that the overall value of the entropy production by direct dissipation for the wall adjacent volume is

$$\boxed{\dot{S}^+_{PRO,\overline{D}\mathrm{mp}} = \frac{1}{2y^+_{\mathrm{mp}}}\left(\frac{1}{2}A_\tau\sqrt{\frac{\pi}{b_\tau}} \cdot \left[\mathrm{erf}\left(\sqrt{b_\tau}\{2y^+_{\mathrm{mp}} - a_\tau\}\right) - \mathrm{erf}\left(-\sqrt{b_\tau}a_\tau\right)\right]\right)} \tag{24}$$

Here the center of the discrete volume has a nondimensional wall-distance y^+_{mp}.

5.1. Entropy production by direct dissipation

The dimensionless form of the entropy production by direct dissipation in the near wall region neglecting wall-

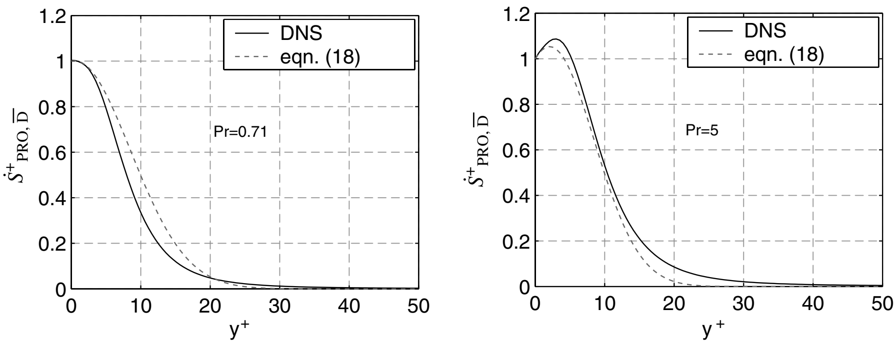

Fig. 3. Entropy production rates by direct dissipation in turbulent heated channel flow from DNS results and Eq. (18), $Ec_\tau = 0.01$, $T_\tau/T_w = 0.01$, $Pr = 0.71$ ($Re_\tau = 395$) and $Pr = 5$ ($Re_\tau = 180$).

5.2. Entropy production by indirect (turbulent) dissipation

The dimensionless form of $\dot{S}_{PRO,D'}$ in the near wall region is

$$\dot{S}^+_{PRO,D'} = \dot{S}_{PRO,D'} \cdot \frac{\nu\alpha\left(\frac{T_w}{T_\tau}\right)^2}{u_\tau^2 \lambda} = \frac{Ec_\tau \frac{T_w}{T_\tau}}{1 + \frac{T_\tau}{T_w}\Theta^+}\varepsilon^+ \tag{25}$$

where $\varepsilon^+ = \varrho\varepsilon\mu/\varrho^2 u_\tau^4$ for small and large values of y^+ is

$$\varepsilon^+(y^+) = \begin{cases} 0.15 & \text{for } y^+ \to 0 \\ 1/(ky^+) & \text{for } y^+ \to \infty \end{cases} \tag{26}$$

Details of this asymptotic behaviour again can be found in [18]. The constant 0.15, however, is not a universal near-wall value but one that is close to the wall-value of ε^+ for different flows over a wide range of Reynolds numbers.

Together with the asymptotic representation of $\Theta^+(y^+)$ we thus get a function that combines the two asymptotes by just patching them at the selected y^+-position $y^+ = 11.6$:

$$\dot{S}^+_{PRO,D'} = \begin{cases} 0.15 Ec_\tau \dfrac{T_w}{T_\tau} & \text{for } y^+ < 11.6 \\ \dfrac{Ec_\tau \dfrac{T_w}{T_\tau}}{Ky^+\left[1 + \dfrac{T_\tau}{T_w}\left(\dfrac{1}{\kappa_\Theta}\log\left(y^+\right) + C^+_\Theta\right)\right]} & \text{for } y^+ \geqslant 11.6 \end{cases} \tag{27}$$

We did not smooth the peak at this position since there are good arguments that for higher Reynolds numbers a relative maximum occurs in the vicinity of our patching point.

Fig. 4 shows how Eq. (27) compares to DNS data for two different Prandtl numbers. Note that there is a tendency for a local maximum around $y^+ = 11.6$ even for the low Reynolds numbers of this case.

This wall function can be integrated so that the overall value of the entropy production by turbulent dissipation for the wall nearest volume is (y^+_{mp}: volume center, $y^+_{\ln\tau} = 11.6$):

$$\boxed{\dot{S}^+_{PRO,D'\mathrm{mp}} = \frac{1}{2y^+_{\mathrm{mp}}}\left[0.15 Ec_\tau \frac{T_w}{T_\tau} y^+_{\ln\tau} + Ec_\tau \frac{T_w^2}{T_\tau^2}\frac{1}{K}\cdot\left[\log\left\{1 + \frac{T_\tau}{T_w}(\log(2y^+_{\mathrm{mp}}) + C^+_\tau) - \log\left(1 + \frac{T_\tau}{T_w}(\log(y^+_{\ln\tau}) + C^+_\tau)\right)\right\}\right]\right]} \tag{28}$$

5.3. Entropy production by mean temperature gradients

$\dot{S}_{PRO,\overline{C}}$ can be handled just like $\dot{S}_{PRO,\overline{D}}$ in Section 5.1. Neglecting wall-parallel temperature gradients the dimensionless form is assumed to be

$$\dot{S}^+_{PRO,\overline{C}} = \dot{S}_{PRO,\overline{C}} \frac{\nu\alpha\left(\frac{T_w}{T_\tau}\right)^2}{u_\tau^2\lambda} = \frac{\left(\frac{d\Theta^+}{dy^+}\right)^2}{Pr\left(1 + \frac{T_\tau}{T_w}\Theta^+\right)^2} = A_\Theta \exp\left[-b_\Theta(y^+ - a_\Theta)^2\right] \tag{29}$$

With $\dot{S}_{PRO,\overline{C}}(y^+ = 0) = Pr$ and the first derivative with respect to y^+ at the wall $\mathrm{d}\dot{S}_{PRO,\overline{C}}/\mathrm{d}y^+(y^+ = 0) = -2Pr^2 T_\tau/T_w$, together with an assumption about the inflection point we get

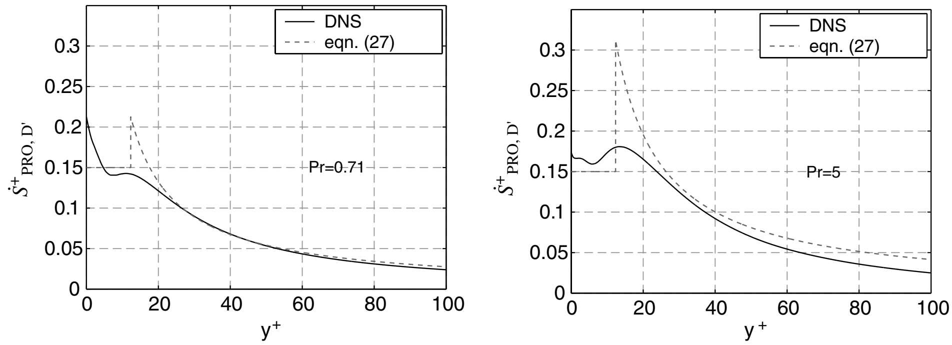

Fig. 4. Entropy production rates by turbulent dissipation in turbulent heated channel flow from DNS results and Eq. (27), $Ec_\tau = 0.01$, $T_\tau/T_w = 0.01$, $Pr = 0.71$ ($Re_\tau = 395$) and $Pr = 5$ ($Re_\tau = 180$).

$$a_\Theta = \frac{-9PrT_\tau y^{+2}_{\ln\Theta}}{4T_w - 12y^+_{\ln\Theta}PrT_\tau + 4\sqrt{T_w^2 - 6y^+_{\ln\Theta}PrT_wT_\tau}} \tag{30}$$

$$b_\Theta = \frac{4T_w - 12y^+_{\ln\Theta}PrT_\tau + 4\sqrt{T_w^2 - 6y^+_{\ln\Theta}PrT_wT_\tau}}{9T_w y^{+2}_{\ln\Theta}} \tag{31}$$

$$A_\Theta = Pr\exp[b_\Theta a_\Theta^2] \tag{32}$$

with $y^+_{\ln\Theta}$ being the edge of the temperature sublayer, i.e $y^+_{\ln\Theta} = 12.1$ for $Pr = 0.71$ and $y^+_{\ln\Theta} = 7.3$ for $Pr = 5$ (these values are determined from the asymptotes of $\Theta^+(y^+)$, see [18]). Fig. 5 shows $\dot{S}_{PRO,\overline{C}}$ for two different Prandtl numbers together with DNS data. Again they compare very well. Integration over the wall nearest volume gives:

$$\boxed{\dot{S}^+_{PRO,\overline{C}\mathrm{mp}} = \frac{1}{2y^+_{\mathrm{mp}}}\left(\frac{1}{2}A_\Theta\sqrt{\frac{\pi}{b_\Theta}}\cdot\left[\mathrm{erf}\left(\sqrt{b_\Theta}\{2y^+_{\mathrm{mp}} - a_\Theta\}\right) - \mathrm{erf}\left(-\sqrt{b_\Theta}a_\Theta\right)\right]\right)} \tag{33}$$

5.4. Entropy production by fluctuating temperature gradients

$\dot{S}^+_{PRO,C'}$ can be handled just like $\dot{S}^+_{PRO,D'}$ in Section 5.2. Its dimensionless form is

$$\dot{S}^+_{PRO,D'} = \dot{S}_{PRO,D'}\cdot\frac{\nu\alpha\left(\frac{T_w}{T_\tau}\right)^2}{u_\tau^2\lambda} = \frac{\varepsilon^+_\Theta}{\left(1+\frac{T_\tau}{T_w}\Theta^+\right)^2} \tag{34}$$

where $\varepsilon^+_\Theta = \varepsilon_\Theta\cdot\nu/(T_\tau^2u_\tau^2)$ for small and large values of y^+ is (details in [18]):

$$\varepsilon^+_\Theta(y^+) = \begin{cases} 0.15Pr & \text{for } y^+\to 0 \\ 1/(K_\Theta y^+) & \text{for } y^+\to\infty\end{cases} \tag{35}$$

With a patching procedure analogous to Eq. (27) we get a distribution for $\dot{S}^+_{PRO,C'}$, which in Fig. 6 is compared to DNS data at $Pr = 0.71$ and $Pr = 5$. For higher Prandtl numbers deviations increase though the wall values are remarkably close together.

For the wall nearest volume integration gives (y^+_{mp}: volume center, $y^+_{\ln\Theta} = 12.1$ for $Pr = 0.71$):

$$\boxed{\dot{S}^+_{PRO,C'\mathrm{mp}} = \frac{1}{2y^+_{\mathrm{mp}}}\left[0.15Pry^+_{\ln\Theta} + \frac{1}{\frac{T_\tau}{T_w} + \log(y^+_{\ln\Theta}) + C^+_\Theta} - \frac{1}{\frac{T_\tau}{T_w} + \log(2y^+_{\mathrm{mp}}) + C^+_\Theta}\right]} \tag{36}$$

5.5. Summary

At this stage all four wall functions are at hand and entropy production in the wall nearest finite volume has been determined as $\dot{S}^+_{PRO,\overline{D}\mathrm{mp}}$ in Eq. (24), $\dot{S}^+_{PRO,D'\mathrm{mp}}$ in Eq. (28), $\dot{S}^+_{PRO,\overline{C}\mathrm{mp}}$ in Eq. (33) and $\dot{S}^+_{PROC'\mathrm{mp}}$ in Eq. (36).

Implementation in a CFD code is straight forward, so that our model of entropy production can be tested for a "real" problem.

6. An Example: entropy production in a heated pipe-flow

Bejan [1] studied the following optimisation problem on the background of a general second law analysis, but without detailed calculation of the entropy production.

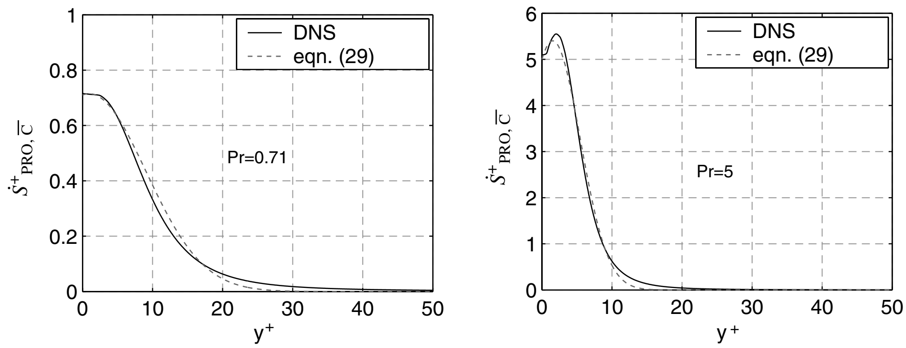

Fig. 5. Entropy production rates by mean temperature gradients in turbulent heated channel flow from DNS results and Eq. (29), $Re_\tau = 395$, $Ec_\tau = -0.01$, $T_w/T_\tau = -0.01$, $Pr = 0.71$ and $Pr = 5$.

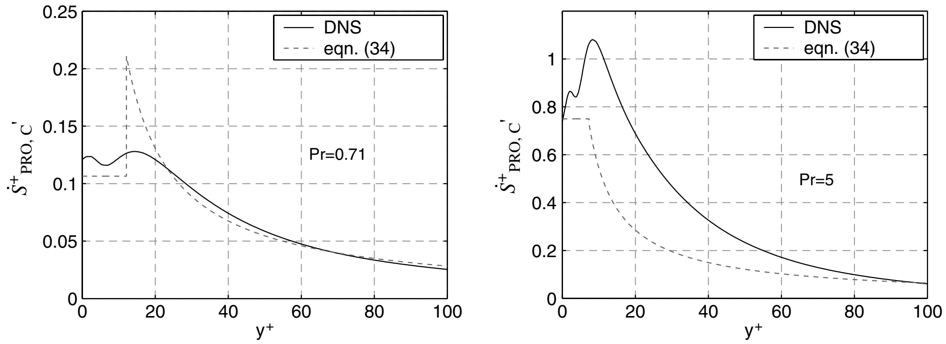

Fig. 6. Entropy production rates by heat transfer over fluctuating temperature gradients in turbulent heated channel flow from DNS results and Eq. (34), $Ec_\tau = 0.01$, $T_\tau/T_w = 0.01$, $Pr = 0.71$ ($Re_\tau = 395$) and $Pr = 5$ ($Re_\tau = 180$).

Instead, for this specific problem he could estimate the overall entropy production from empirical correlations for the friction factor and the Nusselt number. In the example water with a mass flux of 1 kg/s enters a pipe at 300 K and is heated by a constant wall heat flux $q_w = 10^5$ W/m^2 to 310 K at the exit. The turbulent flow is assumed to be hydrodynamically and thermally fully developed. The only adjustable parameter is the pipe diameter. A small diameter corresponds to a long pipe and small temperature differences within a cross-section with small entropy productions by heat transfer but large entropy productions by dissipation. For a large pipe-diameter, however, the pipe is short with large entropy productions by heat transfer but small entropy productions by dissipation. There should be a diameter optimum at which the total entropy production reaches a minimum.

To solve this problem we integrated the specific entropy production rates over the whole pipe volume for different pipe-diameters, i.e. for different Reynolds numbers, $Re = (u_m D)/\nu = (4\dot{m})/(\pi\mu D) \propto 1/D$. For each case ($Re$ fixed) we thus calculated the total entropy production rate, $\dot{S}_{PRO}$, shown in Fig. 7. The minimum of

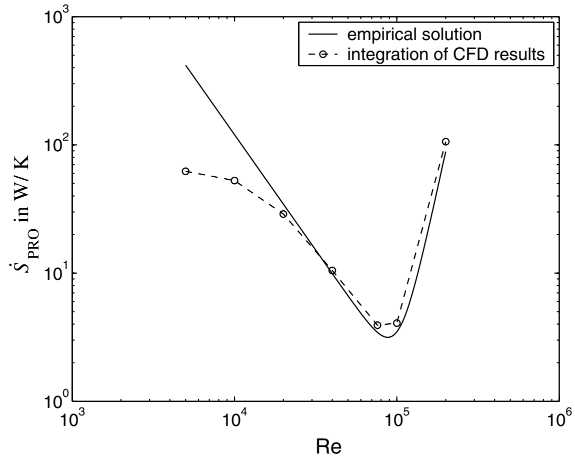

Fig. 7. Total entropy production rate in a turbulent heated pipe flow as a function of Reynolds number.

the entropy production rate as well as the Reynolds number for which it occurs is almost the same for the empirical solution and the presented model equations.

For small Reynolds-numbers the assumptions underlying the empirical correlations approach by Bejan [1] become invalid and larger deviations occur.

At first glance there might emerge the wrong conclusion from the results of this example: Why should one calculate entropy production over the whole flow field when the same result can be achieved by simply exploiting momentum and heat transfer coefficients? Our answer is: the Bejan-approach is good for certain simple cases where cross-sectional distributions of the field quantities are such that global momentum and heat transfer coefficients exist, but not for general complex cases. Then only a detailed information about the field quantity "entropy production" can lead to the total production which may serve as a criterion to find out which of several versions of an apparatus is the best in terms of entropy production.

7. Conclusions/summary

A systematic procedure has been presented to derive formulations for the local entropy production rates in turbulent flows with heat-transfer. The procedure is based on the Reynolds-averaged transport equation for entropy. Four sources of entropy production in turbulent flows with heat transfer can be identified: Entropy production by direct dissipation, by turbulent dissipation, by heat transfer with mean temperature gradients and by heat transfer with gradients of the fluctuating temperature. For each entropy production rate a model equation in combination with the standard k–ε model is derived. It turns out, that peak values of entropy production occur very close to a wall. We therefore introduced semi-empirical wall-functions for the entropy production terms on the basis of asymptotic considerations. The overall model has been "tuned" by results of direct numerical simulations. Its ability to predict the minimum entropy production in a problem of second-law analysis has been shown in an example. The presented model can easyly be implemented in the post-process of a k–ε model CFD analysis and can serve as a powerful tool to calculate efficiency parameters in turbulent flows with heat transfer.

References

[1] A. Bejan, Entropy Generation Minimization, CRC Press, 1996.
[2] A. Bejan, Second law analysis in heat transfer, Energy 5 (1980) 721–732.
[3] A. Bejan, A study of entropy generation in fundamental convective heat transfer, Trans. ASME 101 (1979) 718–725.
[4] A. Bejan, General criterion for rating heat-exchanger performance, Int. J. Heat Mass Transfer 21 (1978) 655–658.
[5] V. Zimparov, Extended performance evaluation criteria for enhanced heat transfer surfaces, Int. J. Heat Mass Transfer 43 (2000) 3137–3155.
[6] M. Sasikumar, C. Balaji, Optimization of convective fins systems: a holistic approach, Heat Mass Transfer 39 (1) (2002) 57–68.
[7] G. Gerdov, Second law analysis of convective heat transfer in flow through a duct with heat flux as a function of duct length, HVAC R Res. 2 (2) (1996) 149–157.
[8] A.Z. Sahin, Second law analysis of laminar viscous flow through a duct subjected to constant wall temperature, Trans. ASME 120 (1998) 76–83.
[9] B.A.K. Abu-Hijleh, W.N. Heilen, Entropy generation due to laminar natural convection over a heated rotating cylinder, Int. J. Heat Mass Transfer 42 (22) (1999) 4225–4233.
[10] B.A. Abu-Hiljleh, M. Abu-Qudais, E.A. Nada, Numerical prediction of entropy generation due to natural convection from a horizontal cylinder, Energy 24 (1999) 327–333.
[11] S.Z. Shuja, B.S. Yilbas, M.O. Budair, I.S. Hussaini, Entropy analysis of a flow past a heat-generated bluff body, Int. J. Energy Res. 23 (1999) 1133–1142.
[12] E. Sciubba, A minimum entropy generation procedure for the discrete pseudo-optimization of finnes-tube heat exchangers, Rev. Gen. Therm. 35 (1996) 517–525.
[13] E. Sciubba, Calculating entropy with cfd, ASME Mech. Eng. 119 (1997) 86–88.
[14] C.Y. Perng, D. Chu, Entropy production and loss evaluation in flow fields. Technical Report 95-WA/HT-13, ASME, 1995.
[15] P.L. Benedetti, E. Sciubba, Numerical calculation of the local entropy generation in the flow around a heated finned tube, Technical Report AES-3, ASME, 1993.
[16] M.K. Drost, M.D. White, Numerical predictions of local entropy generation in an impinging jet, J. Heat Transfer 113 (1991) 823–829.
[17] J.H. Spurk, Strömungslehre, Springer Verlag, 1989.
[18] K. Gersten, H. Herwig, Strömungsmechanik, Vieweg, 1992.
[19] J. Mathieu, J. Scott, An Introduction to Turbulent Flow, first ed., Cambridge University Press, 2000.
[20] Y. Nagano, C. Kim, A two-equation model for heat transport in wall turbulent shear flows, J. Heat Transfer 110 (1988) 583–589.
[21] H. Kawamura, H. Abe, Y. Matsuo, DNS of turbulent heat transfer in channel flow with respect to Reynolds and Prandtl number effects, Int. J. Heat Fluid Flow 20 (1999) 196–207.
[22] S.W. Churchill, R. Usagi, A general expression for the correlation of rates of transfer and other phenomena, AIChE J. 18 (6) (1972) 1121–1128.